

## 1.5 – Elementary Matrices and a Method for Finding $A^{-1}$

Definition 1: Matrices  $A$  and  $B$  are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Definition 2: A matrix  $E$  is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

### **Theorem 1.5.1** Row Operations by Matrix Multiplication

If the elementary matrix  $E$  results from performing a certain row operation on  $I_m$  and if  $A$  is an  $m \times n$  matrix, then the product  $EA$  is the matrix that results when this same row operation is performed on  $A$ .

6. An elementary matrix  $E$  and a matrix  $A$  are given. Identify the row operation corresponding to  $E$  and verify that the product  $EA$  results from applying the row operation to  $A$ .

a.  $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b.  $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c.  $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$



7. Use the following matrices and find an elementary matrix  $E$  that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix},$$
$$D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}, F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

- a.  $EA = B$
- b.  $EB = A$
- c.  $EA = C$
- d.  $EC = A$

**Theorem 1.5.2** Every elementary matrix is invertible, and the inverse is also an elementary matrix.

**Theorem 1.5.3** Equivalent Statements

If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent, that is, all are true or all false.

- a)  $A$  is invertible.
- b)  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- c) The reduced row echelon form of  $A$  is  $I_n$ .
- d)  $A$  is expressible as a product of elementary matrices.



Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

15. 
$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$



$$18. \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$$

19. Find the inverse of each of the following  $4 \times 4$  matrices, where  $k_1, k_2, k_3, k_4, k$  are all nonzero.

a. 
$$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$$

b. 
$$\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

27. Show that the matrices  $A$  and  $B$  are row equivalent by finding a sequence of elementary row operations that produces  $B$  from  $A$ , and then use that result to find a matrix  $C$  such that  $CA = B$ .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

24. Express the matrix and its inverse as products of elementary matrices.

$$\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

33. Prove that if  $B$  is obtained from  $A$  by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to  $B$  recovers  $A$ .